Section 4.3

The Mean Value Theorem

(1) Rolle's Theorem and the Mean Value Theorem

(2) The First Derivative Test

Rolle's Theorem

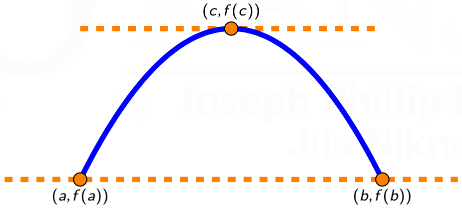
Suppose that f is a function such that

(I) f is continuous on $[a, b]$,

(II) f is differentiable on (a, b),

(III) $f(a) = f(b)$.

Then there exists a value c in (a, b) such that $f'(c) = 0$.



Equivalently: there is a value c in (a, b) such that the tangent line at $(c, f(c))$ is **parallel** to the secant line from $(a, f(a))$ to $(b, f(b))$.

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Equivalently: there is a value c in (a, b) such that the tangent line at $(c, f(c))$ is **parallel** to the secant line from $(a, f(a))$ to $(b, f(b))$.

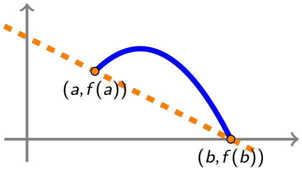
What if we drop the assumption that $f(a) = f(b)$?

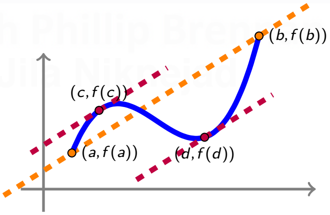
The Mean Value Theorem (MVT)

Suppose that f is a function that is **(I)** continuous on $[a, b]$, and **(II)** differentiable on (a, b).

Then there exists a value c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Idea: Take the secant line from $(a, f(a))$ to $(b, f(b))$ and slide it until it becomes a tangent line.





Mean Value Theorem: Examples

Example 1: Let $f(x) = x^3 - x$ on the interval $[0,2]$.

Solution: The requirements of the Mean Value Theorem are satisfied since polynomials are differentiable (thus continuous) everywhere.

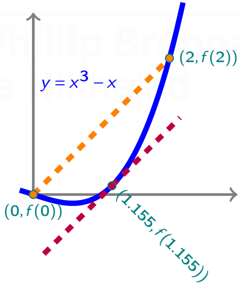
$$f(0) = 0 \qquad f(2) = 6 \qquad \text{Secant line slope: } \frac{f(2)-f(0)}{2-0} = 3$$

The MVT guarantees that there is **some** c in $(0,2)$ such that $f'(c) = 3$.

In this case we can find c explicitly:

$$3 = f'(c) = 3c^2 - 1 \qquad 3c^2 = 4$$

$$\Longrightarrow \quad c = \frac{2}{\sqrt{3}} \approx 1.155$$



Mean Value Theorem: Examples

Example 2: Let $f(x) = x^4 - x^2 - 3x$ on the interval $[0,2]$.

<u>Solution:</u> The requirements of the Mean Value Theorem are satisfied since polynomials are differentiable (thus continuous) everywhere.

$$f(0) = 0 \qquad f(2) = 6 \qquad \text{Secant line slope: } \frac{f(2) - f(0)}{2 - 0} = 3$$

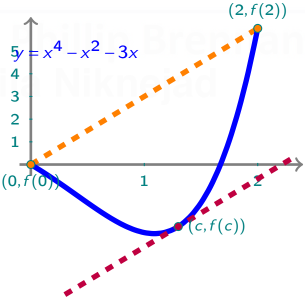
The MVT guarantees that there is **some** c in $(0,2)$ such that $f'(c) = 3$.

Can we find c explicitly?

$3 = f'(c) = 4c^3 - 2c - 3$

$\Rightarrow \quad 4c^3 - 2c - 6 = 0$

No, but we know it must exist!

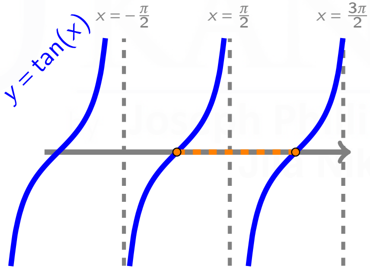


Mean Value Theorem: Examples

Example 3: Let $f(x) = \tan(x)$.

Since $f(0) = f(\pi)$, does Rolle's Theorem guarantee that there is a value c in $(0, \pi)$ such that $f'(c) = 0$?

— **NO**, because $f(x)$ is not continuous on $[0, \pi]$.



(In fact, $f'(x) = \sec^2(x) > 0$ for all x in the domain.)

Mean Value Theorem:Examples

Example 4: Find M and m such that $m \le f(-2) \le M$ if f is a function where $f(1) = 3$ and $-1 \le f'(x) \le 4$ for all x.

Solutions:

Since $f'(x)$ exists everywhere, the Mean Value Theorem applies to $f(x)$ on the interval $[-2, 1]$. There exist Some c in $(-2, 1)$ where

$$f'(c) = \frac{\overset{3}{f(1)} - f(-2)}{1 - (-2)} = \frac{3 - f(-2)}{3} = \textit{Slope of Secant line}$$

$$\Longrightarrow 3f'(c) = 3 - f(-2) \Longrightarrow f(-2) = 3 - 3f'(c)$$

$$\Longrightarrow 3 - 3(4) \le f(c) \le 3 - 3(-1) \Longrightarrow \underbrace{-9}_{m} \le f(-2) \le \underbrace{6}_{M}$$

Consequences of the Mean Value Theorem

Theorem 1: If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Theorem 2: If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).

Theorem 3: If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + C$, where C is some constant.

Proof of Theorem 1: Suppose that A, B are arbitrary numbers with $a < A < B < b$.

Since f' exists for all values in (a, b), f is continuous and differentiable on (a, b) and MVT holds on the interval $[A, B]$.

Therefore, there exists c in $[A, B]$ such that

$$f'(c) = \frac{f(B) - f(A)}{B - A}$$

But $f'(c) = 0$, so $f(B) - f(A) = (B - A)0 = 0$.

Consequences of the Mean Value Theorem

Theorem 1: If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Theorem 2: If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).

Theorem 3: If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + C$, where C is some constant.

Proof of Theorem 2: We need to prove that if $a < A < B < b$, then $f(A) < f(B)$. Again, apply the MVT to the interval $[A, B]$. The conclusion is that there exists c in $[A, B]$ such that

$$f'(c) = \frac{f(B) - f(A)}{B - A}.$$

But $f'(c) > 0$ and $B - A > 0$, so $f(B) - f(A) > 0$, i.e., $f(B) > f(A)$.

Consequences of the Mean Value Theorem

Theorem 1: If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Theorem 2: If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).

Theorem 3: If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + C$, where C is some constant.

Proof of Theorem 3: Let $h(x) = f(x) - g(x)$.

Since $f'(x) = g'(x)$, it follows that $h'(x) = 0$ on (a, b).

Now Theorem 1 implies that $h(x) = C$ on (a, b), where C is some constant.

Therefore $f(x) = g(x) + C$.

The First Derivative Test

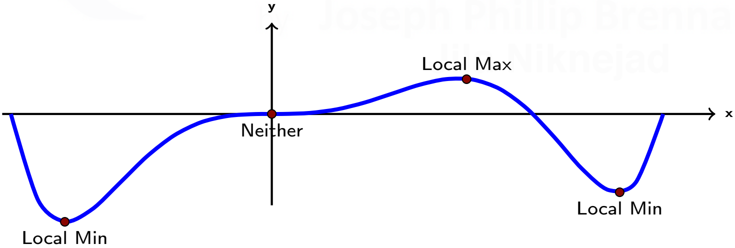
First Derivative Test for Local Extrema

Suppose that c is a critical number of a continuous function f.

(I) If f' changes from positive to negative at c, then f has a local maximum at c.

(II) If f' changes from negative to positive at c, then f has a local minimum at c.

(III) If f' does not change sign at c, then c is not a local extremum.



Example 5: Find the local extrema of f if $f'(x) = (x-4)^3(x+3)^7(x-2)^6$.

Solution: Observe that f has critical numbers $\{-3, 2, 4\}$.

Break the number line into intervals at the critical numbers.

Then determine the sign of $f'(x)$ at some value in each interval.

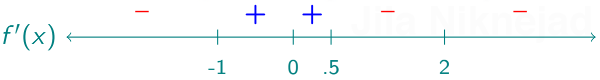
- $(-3, f(-3))$: local maximum point
- $(2, f(2))$: not an extremum
- $(4, f(4))$: local minimum point

Example 6: Find the local extrema of the function $f(x) = \dfrac{(x+1)^2}{x(x-2)}$.

Solution: First, calculate $f'(x)$ using the Quotient Rule:

$$f'(x) = \frac{x(x-2)[2(x+1)] - (x+1)^2[2x-2]}{x^2(x-2)^2}$$
$$= \frac{2(x+1)\big(x(x-2)-(x+1)(x-1)\big)}{x^2(x-2)^2} = \frac{2(x+1)(1-2x)}{x^2(x-2)^2}$$

Critical numbers: $-1, \frac{1}{2}, 0, 2$. Note that 0,2 are not in the domain of f.



By the First Derivative Test, $x = -1$ is a local minimum and $x = \frac{1}{2}$ is a local maximum.

Example 6 (continued): Find the local extrema of the function $f(x) = \dfrac{(x+1)^2}{x(x-2)}$.

By the First Derivative Test, $x = -1$ is a local minimum and $x = \frac{1}{2}$ is a local maximum.